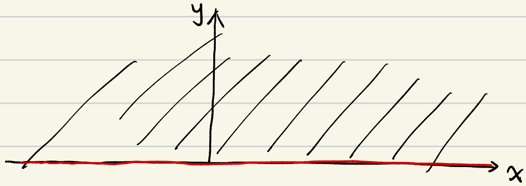
Fourier Analysis April 16, 2024

Review.

Steady state heat equation on the upper half plane.



$u = u(x,y)$ temperation distribution

$$\begin{cases} \Delta u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0, & x \in \mathbb{R},\ y > 0, \\ u(x,0) = f(x), & x \in \mathbb{R}. \end{cases}$$

Thm (Uniqueness)

Let $u \in C^2(\mathbb{R} \times \mathbb{R}_+) \cap C(\overline{\mathbb{R} \times \mathbb{R}_+})$.

Suppose u satisfies that

$$\begin{cases} \Delta u = 0 & \text{on } \mathbb{R} \times \mathbb{R}_+ \\ u(x,0) = 0 & \text{on } \mathbb{R} \end{cases}$$

Moreover suppose that $u(x,y) \to 0$ as $|x| + y \to \infty$.

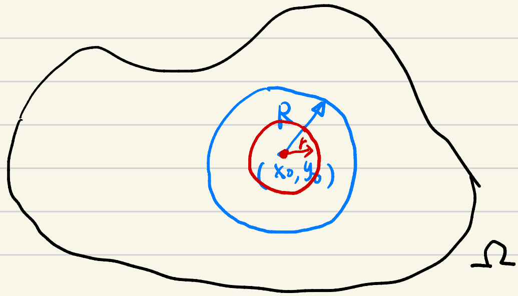
Then $u(x,y) \equiv 0$ on $\mathbb{R} \times \mathbb{R}_+$.

In last class, we proved the above uniqueness result by assuming the following.

Lemma 2 (Mean value property of harmonic function)

Let Ω be an open set of $\mathbb{R}^2$. Let $u \in C^2(\Omega)$ and suppose $\Delta u = 0$ on Ω. Suppose $B_R(x_0, y_0) \subset \Omega$ where

$$B_R(x_0, y_0) = \{ (x,y) \in \mathbb{R}^2 : (x-x_0)^2 + (y-y_0)^2 \leq R^2 \}$$



Then $\forall \ 0 < r < R$,

$$u(x_0, y_0) = \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + r\cos\theta, y_0 + r\sin\theta) \, d\theta$$

Pf. Define

$$U(r,\theta) = u(x_0 + r\cos\theta,\ y_0 + r\sin\theta),$$

for $0 \leq r < R$, $0 \leq \theta < 2\pi$.

Since $\Delta u = 0$, it follows that

$$\Delta U = 0 \quad \text{on } 0 \leq r < R,\ 0 \leq \theta < 2\pi$$

Recall that

$$\Delta U = \frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}$$

Hence

$$r^2 \frac{\partial^2 U}{\partial r^2} + r\frac{\partial U}{\partial r} + \frac{\partial^2 U}{\partial \theta^2} = 0$$

so

$$r\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right) + \frac{\partial^2 U}{\partial \theta^2} = 0$$

Define $f(r) = \frac{1}{2\pi}\int_0^{2\pi} U(r,\theta)\, d\theta$

$$r\frac{d}{dr}\left(r\frac{df(r)}{dr}\right) = \frac{1}{2\pi}\int_0^{2\pi} r\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right) d\theta$$

$$= -\frac{1}{2\pi}\int_0^{2\pi} \frac{\partial^2 U}{\partial \theta^2}\, d\theta$$

$$= -\frac{1}{2\pi} \left. \frac{\partial U}{\partial \theta} \right|_0^{2\pi}$$

$$= 0 \qquad \left(\text{since } \frac{\partial U}{\partial \theta} \text{ is } 2\pi\text{-perodic in } \theta \right)$$

Hence

$$\frac{d}{dr}\left(r \frac{df}{dr} \right) \equiv 0 \quad \text{on} \quad 0 < r < R$$

So $\quad r \dfrac{df}{dr} \equiv \text{Const} \quad$ on $\quad (0, R)$

Notice that as $r \to 0$, $\dfrac{df}{dr}$ is bdd, hence

$$r \frac{df}{dr} = 0$$

It implies $f \equiv \text{Const}$ on $0 < r < R$

By continuity $f(r) \equiv f(0)$.

That is

$$\frac{1}{2\pi} \int_0^{2\pi} \underbrace{U(r,\theta)}_{u(x_0 + r\cos\theta,\ y_0 + r\sin\theta)} d\theta = \frac{1}{2\pi} \int_0^{2\pi} U(0,\theta)\, d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} u(x_0, y_0)\, d\theta$$

$$= u(x_0, y_0). \qquad \blacksquare$$

§5.3 Poisson summation formula.

Thm 3 (Poisson summation formula).

Let $f \in \mathcal{M}(\mathbb{R})$. Assume that $\hat{f} \in \mathcal{M}(\mathbb{R})$.

Let $F(x) = \sum_{n\in\mathbb{Z}} f(x+n)$.

Then

$$F(x) = \sum_{n\in\mathbb{Z}} \hat{f}(n)\, e^{2\pi i n x}$$

In particular,

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n).$$

Pf. Since $f \in \mathcal{M}(\mathbb{R})$, $\exists$ a constant $C > 0$ such that

$$|f(x+n)| \leq \frac{C}{1+(x+n)^2}.$$

It follows that

$$\sum_{n=-N}^{N} f(x+n) \rightrightarrows F(x) \quad \text{on any compact set,}$$

as $N \to \infty$.

So F is cts on $\mathbb{R}$, and it is 1-periodic

Now let us calculate the Fourier coefficients of F.

$$\hat{F}(n) = \int_0^1 F(x) e^{-2\pi i n x} dx$$

$$= \int_0^1 \sum_{k=-\infty}^{\infty} f(x+k) \cdot e^{-2\pi i n x} dx$$

By the (DCT)

$$= \sum_{k=-\infty}^{\infty} \int_0^1 f(x+k) e^{-2\pi i n x} dx$$

$$= \sum_{k=-\infty}^{\infty} \int_k^{k+1} f(y) e^{-2\pi i n (y-k)} dy$$

$$= \sum_{k=-\infty}^{\infty} \int_k^{k+1} f(y) \cdot e^{-2\pi i n y} \, dy$$

$$= \int_{-\infty}^{\infty} f(y) e^{-2\pi i n y} \, dy$$

$$= \hat{f}(n)$$ ($\leftarrow$ Fourier transform of f)

That is, $\hat{F}(n) = \hat{f}(n)$, $n \in \mathbb{Z}$.

Since $\hat{f} \in M(\mathbb{R})$,

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty.$$

It follows that

$$\sum_{n=-\infty}^{\infty} |\hat{F}(n)| < \infty$$

Recall $\hat{F}$ is the Fourier coefficient of F on $[0,1]$.

So
$$F(x) = \sum_{n=-\infty}^{\infty} \hat{F}(n) e^{2\pi i n x} \quad \text{on} \quad [0,1].$$

That is,
$$F(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n x}.$$

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Example 1. Let $f(x) = P_y(x) = \frac{1}{\pi} \frac{y}{x^2+y^2}$ on $\mathbb{R}$,

where $y>0$.

Recall that
$$\hat{f}(\xi) = e^{-2\pi|\xi|y}.$$

Then $f, \hat{f} \in \mathcal{M}(\mathbb{R})$. By the Poisson Summation formula,
$$\sum_{n=-\infty}^{\infty} f(x+n) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n x}$$
$$= \sum_{n\in\mathbb{Z}} e^{-2\pi|n|y} e^{2\pi i n x}$$

$$= P_r(2\pi x)$$

where $r = e^{-2\pi y}$ and

$$P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta} = \frac{1-r^2}{1-2r\cos\theta + r^2}.$$

Hence

$$\sum_{n\in\mathbb{Z}} \mathcal{P}_y(x+n) = P_{e^{-2\pi|y|}}(2\pi x).$$

Relation between the Poisson kernel on the upper half plane and the Poisson kernel on the unit disc.

Example 2. Define the Theta function

$\Theta : \mathbb{R}_+ \to \mathbb{R}$ by

$$\Theta(s) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 s}.$$

Then Θ satisfies the following identity

$$\Theta(s) = \frac{1}{\sqrt{s}}\Theta\left(\frac{1}{s}\right).$$

Pf. Fix $s>0$. Set $f(x) = e^{-\pi x^2 s}$.

Notice that

$$e^{-\pi x^2} \xrightarrow{\mathcal{F}} e^{-\pi \xi^2}.$$

So

$$f(x) = e^{-\pi(\sqrt{s}\cdot x)^2} \xrightarrow{\mathcal{F}} \frac{1}{\sqrt{s}} e^{-\pi\left(\frac{\xi}{\sqrt{s}}\right)^2}$$

$$= \frac{1}{\sqrt{s}} e^{-\pi \xi^2/s}.$$

That is, $\hat{f}(\xi) = \frac{1}{\sqrt{s}}\cdot e^{-\pi\xi^2/s}.$

Check: $f, \hat{f} \in \mathcal{M}(\mathbb{R})$.

By Poisson summation formula,

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n),$$

Hence

$$\sum_{n\in\mathbb{Z}} e^{-\pi n^2 s} = \sum_{n\in\mathbb{Z}} \frac{1}{\sqrt{s}} e^{-\pi \frac{n^2}{s}}$$

i.e.

$$\Theta(s) = \frac{1}{\sqrt{s}} \Theta\left(\frac{1}{s}\right). \quad \square$$

Example 3. Relation between the heat kernels on the line and the circle:

$$\mathcal{H}_t(x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}, \quad x\in\mathbb{R}, \quad t>0$$

(Heat kernel on the real line)

Notice that

$$\widehat{\mathcal{H}_t}(\xi) = e^{-4\pi^2\xi^2 t}$$

Applying the Poisson summation formula to $\mathcal{H}_t(x)$ gives

$$\sum_{n\in\mathbb{Z}} \mathcal{H}_t(x+n) = \sum_{n\in\mathbb{Z}} \widehat{\mathcal{H}_t}(n)\, e^{2\pi i n x}$$

$$= \underbrace{\sum_{n\in\mathbb{Z}} e^{-4\pi^2 n^2 t}\, e^{2\pi i n x}}_{=:\, H_t(x)}$$

(heat kernel on the circle)

Hence

$H_t(x) > 0$ for all $x \in \mathbb{R}$.